Borel ideals

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Let $\mathbb{N}$ be the set of all natural numbers, $[\mathbb{N}]^2$ the set of all unordered pairs of natural numbers, $[\mathbb{N}]^{<\mathbb{N}}$ the set of all finite subsets of natural numbers and $[\mathbb{N}]^\mathbb{N}$ the set of all infinite subsets of natural numbers.

**Theorem (Ramsey)**

For every coloring of pairs of natural numbers i.e. for every $c : [\mathbb{N}]^2 \to 2$, there is an infinite monochromatic subset i.e. infinite $y \subseteq \mathbb{N}$ such that $c \upharpoonright [y]^2$ is constant.
Proof.

We describe an ”algorithm” that produces the desired monochromatic set. We construct inductively \((n_0, x_0), (n_1, x_1), \ldots\) where \(n_{k+1} \in x_n\) and \(x_{k+1} \subseteq x_k\).

Set \(x_0 = \mathbb{N}\) and \(\alpha = 0\) and assume that we are in the \(k + 1\)-th step.

1. If there is an element \(n \in x_k\) such that the set \(y = \{m \in x_k : c(\{n, m\}) = \alpha\}\) is infinite then put \(n_{k+1}\) to be minimal such \(n\) and \(x_{k+1} = y\).

2. If no such element exists put \(x_0 = x_k\), \(\alpha = 1\) and repeat the whole process.

In the end the set \(y = \{n_0, n_1, n_2, \ldots\}\) is as desired. \(\square\)
Important observation is that our "algorithm" works for every infinite $x \subseteq \mathbb{N}$ i.e. if we put $x_0 = x$ we obtain some infinite monochromatic subset $z \subseteq x$.

Therefore for given coloring $c : [\mathbb{N}]^2 \to 2$ our "algorithm" describes a function

$$S_c : [\mathbb{N}]^\mathbb{N} \to [\mathbb{N}]^\mathbb{N}$$

such that

- $S_c(x) \subseteq x$,
- $S_c(x)$ is $c$–monochromatic.

**Observation**

The function $S_c$ is Borel i.e. nicely definable.

One possible interpretation of this Observation is that Ramsey's Theorem has a "Borel proof".
MetaQuestion
Do other Ramsey statements have "Borel proof"?
Recall that there is a topology on $[\mathbb{N}]^\mathbb{N}$ which is completely metrizable and separable i.e. Polish topology. The basic open sets are given as

$$\mathcal{N}_{s,t} = \{ x \in [\mathbb{N}]^\mathbb{N} : s \subseteq x, t \cap x = \emptyset \}$$

where $s, t \in [\mathbb{N}]^{<\mathbb{N}}$. The Borel sets are elements of the smallest $\sigma$–algebra containing the open sets. Function $f : [\mathbb{N}]^\mathbb{N} \to [\mathbb{N}]^\mathbb{N}$ is Borel if pre-images of Borel sets are Borel.
Let $s \in [\mathbb{N}]^{<\mathbb{N}}$ and $t \in [\mathbb{N}]^{<\mathbb{N}}$. We write $s \sqsubseteq t$ when there is $n \in \omega$ such that $s = t \cap \{0, 1, \cdots, n\}$ and we say that $s$ is an initial segment of $t$.

**Theorem (Galvin)**

Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ and $x \in [\mathbb{N}]^{\mathbb{N}}$. Then there is an infinite $y \subseteq x$ such that one of the following holds

- for all $z \in [y]^{\mathbb{N}}$ there is $s \in \mathcal{F}$ such that $s \sqsubseteq z$,
- $[y]^{<\mathbb{N}} \cap \mathcal{F} = \emptyset$. 

A special type of subsets of \([\mathbb{N}]^{<\mathbb{N}}\) is as follows. We say that \(B \subseteq [\mathbb{N}]^{<\mathbb{N}}\) is a \textit{front} if

- every two elements of \(B\) are \(\sqsubseteq\)-incomparable,
- every \(x \in [\mathbb{N}]^{\mathbb{N}}\) has an initial segment in \(B\).

\textbf{Theorem (Nash-Williams)}

Let \(B\) be a front on \(\mathbb{N}\) and \(\mathcal{F} \subseteq B\). Then for every infinite \(x \in [\mathbb{N}]^{\mathbb{N}}\) there is an infinite \(y \subseteq x\) such that one of the following holds

- \([y]^{<\mathbb{N}} \cap B \subseteq \mathcal{F}\),
- \([y]^{<\mathbb{N}} \cap \mathcal{F} = \emptyset\).
Both Theorems are just a special case of the following.

**Theorem (Silver)**

Let \( \mathcal{X} \subseteq [\mathbb{N}]^\mathbb{N} \) be an analytic set. Then for every \( x \in [\mathbb{N}]^\mathbb{N} \) there is an infinite \( y \subseteq x \) such that one of the following holds

- \( [y]^\mathbb{N} \subseteq \mathcal{X} \),
- \( [y]^\mathbb{N} \cap \mathcal{X} = \emptyset \).

Let \( \mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}} \) as in the Galvin’s Theorem. Then we put

\[
\mathcal{X}_\mathcal{F} = \{ x \in [\mathbb{N}]^\mathbb{N} : \exists s \in \mathcal{F} \ s \subseteq x \}
\]

and observe that \( \mathcal{X}_\mathcal{F} \) is open subset of \( [\mathbb{N}]^\mathbb{N} \), if moreover \( \mathcal{F} \) is as in the Nash-Williams’s Theorem then \( \mathcal{X}_\mathcal{F} \) is clopen.
Theorem (with C.Uzcategui)

Let $\mathcal{B}$ be a front and $\mathcal{F} \subseteq \mathcal{B}$. There is a Borel map $S : [\mathbb{N}]^\mathbb{N} \to [\mathbb{N}]^\mathbb{N}$ such that $S(x) \subseteq x$ and $S(x)$ satisfies the conclusion of the Nash-Williams’s Theorem.

Theorem (with C.Uzcategui)

There is $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ such that there is no Borel function $S : [\mathbb{N}]^\mathbb{N} \to [\mathbb{N}]^\mathbb{N}$ such that $S(x) \subseteq x$ and $S(x)$ satisfies the conclusion of the Galvin’s Theorem.

In other words in the Ramsey context ”open sets are more complicated then clopen sets”.
The proof of the first Statement uses the fact that every front $B$ has a rank and the proof goes by induction on ranks.

The proof of the second Statement uses a result from the theory of Borel ideals on $\mathbb{N}$.

**Definition**
A family $\mathcal{I} \subseteq [\mathbb{N}]^\mathbb{N}$ is called an ideal on $\mathbb{N}$ if the following holds
- if $x, y \in \mathcal{I}$ then $x \cup y \in \mathcal{I}$,
- if $y \subseteq x$ and $x \in \mathcal{I}$ then $y \in \mathcal{I}$. 
We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ is tall if for every $x \in [\mathbb{N}]^{\mathbb{N}}$ there is $y \in \mathcal{I}$ such that $y \subseteq x$.

**Definition**

Let $\mathcal{I}$ be a tall ideal on $\mathbb{N}$. We say that $\mathcal{I}$ has a Borel selector, if there is a Borel function $S : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ such that for every $x \in [\mathbb{N}]^{\mathbb{N}}$

1. $S(x) \subseteq x$,
2. $S(x) \in \mathcal{I}$.

There is a correspondence between "some" $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ i.e. instances of Galvin’s Theorem and tall $F_\sigma$ ideals on $\mathbb{N}$. Moreover Borel selector for ideal is equivalent to Borel proof of the corresponding instance of Galvin’s Theorem.

**Theorem (with C.Uzcategui)**

There is a $F_\sigma$ tall ideal without a Borel selector.